Semiclassical description of non magnetic cloaking

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Abstract: We present a semiclassical description of non-magnetic cloaking. The semiclassical result is confirmed by numerical simulations of a gaussian beam scattering from the cloak. Further analysis reveals that certain beams penetrate the non-magnetic cloak thereby degrading the performance.

OCIS codes: (160.1190) Anisotropic optical materials; (230.0230) Optical devices; (260.2110) Electromagnetic theory.

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1. Introduction

Recently a lot of interest has been focussed on the phenomenon of cloaking, which deals with hiding objects from electromagnetic fields [1-10]. The metamaterial cloak, designed using the coordinate free representation of Maxwell's equations [2] was experimentally demonstrated at microwave frequencies [3]. While a straightforward extension of this device to the optical regime is problematic due to the difficulities in achieving a magnetic response at high frequencies, the issue was addressed with a design that relies on dielectric anisotropy [4]. However, due to the strict requirements on the electromagnetic response [2] of the medium forming the device, a broadband cloak is difficult to achieve. Furthermore alternate implementations of cloaking rely on resonances [5, 6] making them sensitive to losses. An analysis of the cloaking performance relies on full wave numerical techniques [7] but it is desirable to obtain an intuitive approach for optimiziation and understanding sensitivity of the device to various parameters.

We present a description of the non magnetic cloak using the semiclassical approximation which can aid in a better design and identify the regimes of operation of the device. The obtained qualitative picture based on a Hamiltonian formulation is valid beyond ray optics and helps to identify the non resonant behaviour in the short wavelength limit. As in Ref. [8] which describes ray tracing for the ideal cloak, our approach provides an insight into not only the transformation optics based non magnetic design but also the ray trajectories in the non magnetic cloak, in particular, the smooth bending of rays around the cloaked region. We confirm the semiclassical result by numerical simulations of gaussian beam scattering from the cloak which reveal the bending of the gaussian beam around the cloaked region. Further analysis reveals that an on-axis ray penetrates the cloak and this degradation of cloaking performance due to the on axis gaussian beam is evident in the simulations. Finally we identify that the origin of the unique isotropic scattered intensity present even in a reflectionless design [9] is essentially due to this on axis component of the incident radiation.

2. Ray dynamics of cloaking

Similar to Leonhardt [10], we first look for a dielectric medium which guides ray trajectories around a certain region of space (cloaked region) where the object is placed. The distortion in the ray trajectories as compared to that of vacuum is only local to this cloaked region. Thus the rays which reach the far field carry no information about the object and appear as if traveling in vacuum. Finally we construct a finite sized device from this cloaking medium taking into account refraction at the device boundaries. This leads to an alternate approach to the non magnetic cloak which was designed using transformation optics. We start with the ray-optic Hamiltonian, derived using the semiclassical approximation for a non-magnetic cylindrically anisotropic inhomogeneous medium

$$H = c\sqrt{\frac{p_r^2}{\varepsilon_\theta} + \frac{p_\theta^2}{r^2\varepsilon_r}} \tag{1}$$

where c is the velocty of light in vacuum, p_r, p_θ are the radial and angular momentum and ε_θ, ε_r are the tangential and radial dielectric permittivities. The Hamiltonian can be derived using a procedure outlined in [11], for a non-magnetic medium with fields which have the magnetic field along the axis of the cylinder. Note that the Hamiltonian is related to the dispersion relation in the medium for the appropriate polarization.

2.1. Ray trajectory in vacuum

The direction of the ray is associated with the direction of energy flow determined by the Poynting vector ($\overrightarrow{S}$). In an isotropic medium, a ray is directed along the wavevector ($\overrightarrow{k}$) since the Poynting vector is parallel to $\overrightarrow{k}$. This is not true in the case of an anisotropic medium where $\overrightarrow{k}$ is not necessarily parallel to $\overrightarrow{S}$. The ray trajectory in vacuum is simply a straight line, given by the equation

$$r\sin(\theta) = \rho \tag{2}$$

where the angle θ and distance r has been defined in Fig. 1(b) (inset) and ρ is the impact parameter defined with respect to the origin. This can be viewed as a statement of conservation of angular momentum for the ray and can also be obtained from Hamilton's equations using Eq. 1 and taking $\varepsilon_r = 1$ and $\varepsilon_\theta = 1$.

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2.2. *Ray trajectory in a cloaking medium*

As mentioned earlier we are seeking a medium which bends rays smoothly around a certain region of space and sets the rays back on the path it would take in vacuum. We let this region of space, the cloaked region, be a circular region of radius a embedded in a medium with some effective dielectric permittivity ε_r and ε_θ. For rays which avoid this region of radius a, the equation of the ray trajectory is given as

$$(r-a)\sin(\theta) = \rho \tag{3}$$

We note that rays that start with a positive impact parameter (which is the case for all rays except the ones that pass through the center) would avoid this region. Far away from the cloaked region i.e. when $r >> a$, the rays will essentially travel in straight lines just like in vacuum. Also note that Eq. 3 implies rays with a large impact parameter essentially travel in straight lines just like vacuum. As long as the equation of ray trajectories is given by Eq. 3 the distortion in the ray trajectory as compared to vacuum is only local to the region with radius a about the origin. Thus the rays in this medium have avoided the cloaked region of radius a and appear like traveling in vacuum to a distant observer which is exactly the property of a cloak. The behavior of light in such a cloaking medium has been elucidated in the inset of Fig. 1(b), where the dotted red line is the trajectory in vacuum and the blue line is a trajectory which avoids a region of radius a, obtained by Eq. 3 in the cloaking medium.

2.3. *Material parameters for cloaking*

It remains to be seen what ε_r and ε_θ would lead to trajectories given by Eq. 3 for rays with varying impact parameters ρ. We start by observing that the ray trajectories given by Eq.2 belong to the Hamiltonian for vacuum

$$H_{vacuum} = c\sqrt{p_r^2 + \frac{p_\theta^2}{r^2}} \tag{4}$$

Therefore, we assert that the ray trajectories given by Eq. 3 should belong to the Hamiltonian of the cloaking medium given by

$$H_{cloak} = c\sqrt{p_r^2 + \frac{p_\theta^2}{(r-a)^2}} \tag{5}$$

where a is the radius of the region being cloaked. We note that the cloaking medium should be invariant with respect to θ because rays in any direction should avoid the circular region. The simplest choice of ε_r and ε_θ that gives a Hamiltonian of the form as in Eq. 5 can be obtained by observing the form of the Hamiltonian for a general cylindrically anisotropic medium as in Eq. 1. In particular, the product $r^2\varepsilon_r$ present in the denominator of one of the terms in Eq. 1, should yield $(r-a)^2$ as in Eq. 5, which can be achieved by

$$\varepsilon_r \propto \frac{(r-a)^2}{r^2} \qquad \varepsilon_\theta = 1 \tag{6}$$

Therefore in a medium with the above mentioned dielectric parameters rays of light would smoothly bend around a region of radius a and also appear to a far away observer as traveling in vacuum.

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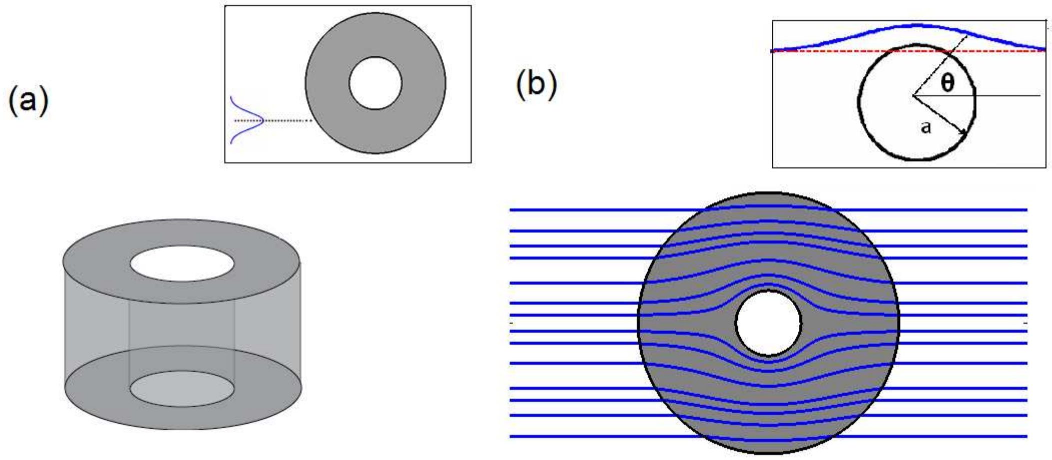

Fig. 1. (a) Cylindrical cloak, gaussian beam incident on the cloak (inset). (b) Calculated ray trajectories. Shown in the inset are ray trajectories in a cloaking medium (blue), ray trajectory in vacuum (red), cloaked region of radius a

2.4. Cloaking device

In any finite sized device (inner radius a and outer radius b), refraction at the boundary between the device and vacuum will have to be taken into consideration for the smooth bending of the rays around the cloaked region. Reflections will also have to be completely avoided or minimized by matching the impedance on the outer surface of the cloak. Retaining the functional form of the cloaking medium dielectric permittivities, we construct the cloaking device using

$$\varepsilon_r = c_1 \frac{(r-a)^2}{r^2} \qquad \varepsilon_\theta = c_2 \tag{7}$$

where c_1 and c_2 are constants to be determined. The cloak Hamiltonian now becomes

$$H_{cloak} = c\sqrt{\frac{p_r^2}{c_2} + \frac{p_\theta^2}{(r-a)^2 c_1}} \tag{8}$$

Solving Hamilton's equations for a ray impinging on the cloak with impact parameter $\rho = b\,\sin(\theta_i)$, where θ_i is the incident angle, we obtain the equation of the ray trajectory inside the cloak as

$$r(\theta) = a + \frac{b\,\sin(\theta_i)}{\sqrt{c_1}\sin\left[\sqrt{\frac{c_2}{c_1}}(\theta - \theta_0)\right]} \tag{9}$$

θ_0 is a constant evaluated by conserving angular momentum at the point of impact of the ray with the cloak.

$$\theta_0 = \theta_i - \sqrt{\frac{c_2}{c_1}}\arcsin\left[\frac{b\,\sin(\theta_i)}{(b-a)\sqrt{c_1}}\right] \tag{10}$$

c_1 and c_2 have to be chosen such that the equation of every ray striking the cloak would be of the form as in Eq. 3. For that we require, from Eq. 9.

$$\theta_0 = 0 \qquad c_1 = c_2 \tag{11}$$

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Substituting this in Eq. 10 gives

$$c_1 = \frac{b^2}{(b-a)^2} \tag{12}$$

So finally the dielectric permittivity of the non magnetic cloak takes the form

$$\varepsilon_r = \eta \frac{(r-a)^2}{r^2} \qquad \varepsilon_\theta = \eta \tag{13}$$

where $\eta = \frac{b^2}{(b-a)^2}$ is a constant related to the dimensions of the cloak. We note that the dielectric permittivities obtained by the semiclassical argument is exactly same as the reduced parameters obtained by the coordinate transformation approach [3, 4]. This implies the cloaking achieved by this medium is valid beyond just ray optics. Note that the cloak with reduced parameters is not perfect but the there are reflections arising essentially due to the tangential permittivity which is different from unity.

3. Numerical simulations

While the tangential permittivity is constant, the radial dielectric response varies from 1 on the outer surface ($r = b$) to 0 on the inner surface ($r = a$) within which the object is placed. Here we try to understand how a gaussian beam scattering from the cloak compares with the above presented semiclassical ray trajectories. As in [11] and references therin, we represent the incident gaussian beam in a basis of angular momentum modes and represent the scattered field as a sum of outgoing cylindrical waves only. If the reflection coefficient of the cloak is known, the transmitted and reflected fields can then be obtained by using the continuity of B_z and $\varepsilon_\theta^{-1}\partial B_z/\partial r$ where the z-axis is directed along the axis of the cylinder.

3.1. Reflection coefficient of the cloak

To calculate the fields in the non-magnetic cloak, the reflection coefficient has to be obtained with high accuracy for all the angular momentum modes interacting with the cloak. We begin by evaluating the transfer matrix for the fields from the outer surface of the cloak to the inner surface. The transfer matrix for the m^{th} angular momentum mode (cylindrical coordinates) is written for the tangential component of the electric field E_θ and the normal magnetic field B_z

$$B_z = R_1^{'}(r) exp(im\theta) \tag{14}$$

$$E_\theta = R_2^{'}(r) exp(im\theta) \tag{15}$$

Solving Maxwell's equations for the polarization of interest, we see that the radial part of the axial magnetic field, $R_1^{'}(r)$, in a cylindrical anisotropic medium satisfies the following equation,

$$r^2 \frac{d^2 R_1^{'}}{dr} + r \frac{dR_1^{'}}{dr} + \left[\varepsilon_\theta k_0^2 r^2 - m^2 \frac{\varepsilon_\theta}{\varepsilon_r}\right] R_1^{'} = 0 \tag{16}$$

For the non-magnetic cloak with parameters as in Eq. 13 the above equation becomes

$$\frac{d^2 R_1^{'}}{dr} + \frac{1}{r}\frac{dR_1^{'}}{dr} + \left[\eta k_0^2 - \frac{m^2}{(r-a)^2}\right] R_1^{'} = 0 \tag{17}$$

From Maxwell's equations we also find

$$R_2^{'}(r) = \frac{1}{ik_0 \varepsilon_\theta} \frac{\partial R_1^{'}(r)}{\partial r} \tag{18}$$

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where $k_0 = \omega/c$. Note that exterior to the cloak and in the cloaked inner region filled with vacuum, the radial part of the axial magnetic field, $R_1(r)$, satisfies the standard Bessel equation and hence we can write the incident, transmitted (into the cloaked region) and scattered field in terms of Hankel functions (H_m^+, H_m^-) where $+$ corresponds to an outgoing cylindrical wave and $-$ to an incoming cylindrical wave.

For $r \geq b$ we have,

$$R_1(r) = H_m^-(k_0 r) + r_m^+ H_m^+(k_0 r) \tag{19}$$

$$R_2(r) = \frac{1}{i}(DH_m^-(k_0 r) + r_m^+ DH_m^+(k_0 r)) \tag{20}$$

and for $0 \leq r \leq a$

$$R_1(r) = t_m^- H_m^-(k_0 r) + t_m^+ H_m^+(k_0 r) \tag{21}$$

$$R_2(r) = \frac{1}{i}(t_m^- DH_m^-(k_0 r) + t_m^+ DH_m^+(k_0 r)) \tag{22}$$

where $H_m^-(k_0 r)$ denotes the radial part of the incident wave, $DH_m(k_0 r)$ denotes the derivative of the Hankel function with respect to the argument, r_{m^+} is the reflection coefficient, t_m^+ and t_m^- are the transmitted outgoing and incoming waves coefficients. The radial functions just outside the cloak and just inside the cloaked region can be related by

$$\begin{pmatrix} R_1(a) \\ R_2(a) \end{pmatrix} = T_m \begin{pmatrix} R_1(b) \\ R_2(b) \end{pmatrix} \tag{23}$$

where T_m is the transfer matrix for the m^{th} angular momentum mode, evaluated numerically. A good check for the accuracy of the evaluated transfer matrix is the fact that

$$\det(T_m) = \frac{r_{\max}}{r_{\text{in}}} \tag{24}$$

which follows from the cylindrical nature of basis chosen. The numerical procedure to evaluate the transfer matrix has to be chosen wisely due to the rapid variation in dielectric permittivities and since all the information about the cloak's response to each angular momentum mode is contained within the constants r_m and t_m which therefore need to be evaluated accurately. To avoid the singularity at the origin inherent in our hankel function basis we demand that

$$t_m^+ = t_m^- \tag{25}$$

Using equations (23) and (25) we can solve for the reflection coefficient and thereby obtain the fields inside and outside the cloak.

4. Results and discussion

4.1. Gaussian beam

We consider a cloak with inner radius $r_{min} = 1.1\lambda$, outer radius $r_{max} = 5.1\lambda$, $\lambda = 365$nm and use the functional form of dielectric permittivities as in Eq. 13. The radial permittivity varies from 1 on the outer interface to 0 on the inner, while the tangential permittivity is constant and different from 1. Rays striking the cloak can be represented as a gaussian beam scattering from the cloak with some impact parameter p. From the simulation result in Fig. 2(b) it is evident that the beam bends around the inner hollow region of the cloak and proceeds in the same direction it is incident from. For comparison we also show a gaussian beam traveling in vacuum in Fig. 2(a). There are reflections at each interface due to the reduced parameter implementation of the cloak. Losses and reflections are the main source of degradation of performance of the non

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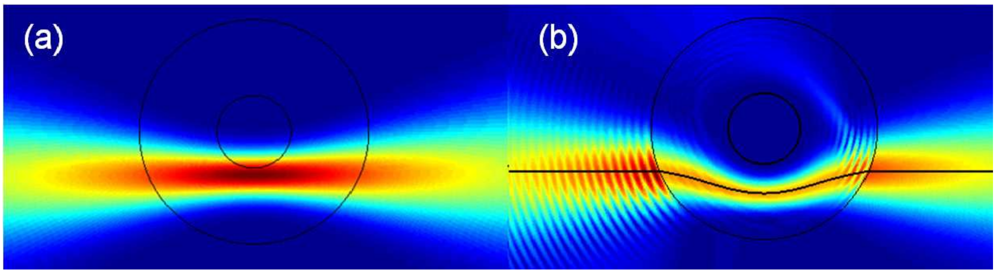

Fig. 2. (Intensity in false color) (a) gaussian beam in vacuum (position of cloak is also shown for comparison) (b) gaussian beam incident on the cloak showing the smooth bending and some reflections. Superimposed ray trajectory in black evaluated analytically corresponds well to the center of the beam.

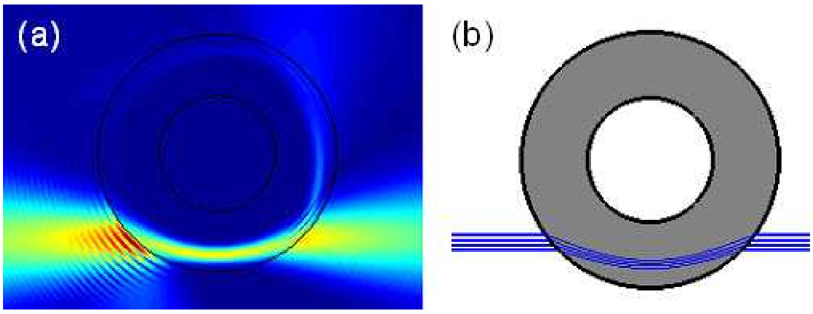

Fig. 3. (a) Intensity in false color of a gaussian beam incident on the cloak showing compression within the device (b) Ray calculations exhibiting the same compression

magnetic cloak [7]. These reflections, which arise due to the tangential permittivity ($\neq 1$) can be reduced by choosing a thick cloak and small cloaked region ($h = b - a \approx b$). Shown also in the figure is the ray trajectory evaluated using the analytical expression. The excellent agreement of the ray trajectory and the path of the center of the gaussian beam validates the semiclassical description.

4.2. Compression

The coordinate transformation approach used in [2] to construct the cloak compresses points from the inner region ($r < a$) to the annulus ($a < r < b$). This compression is thus evident in the field of a gaussian beam being bent around the inner region [Fig. 3(a)]. Ray calculations using the semiclassical approach show the same behaviour [Fig. 3(b)].

4.3. On-axis ray

It is necessary to note that in the ray picture, the on axis ray which comes in with zero angular momentum is not bent around the inner cloaked region. The numerical wave simulation indeed shows that for an on axis gaussian beam the field penetrates inside the cloak. The intensity is shown in Fig. 4(a). Thus an object kept inside the cloak would scatter and be detectable if the probe beam is on axis. Other than the reflections due to the tangential permittivity mismatch, an isotropic scattered pattern is noticed clearly in the real part of the field plotted in Fig. 4(b). This is because the penetrated field escapes the inner cloaked region only in the radial direction with zero angular momentum.

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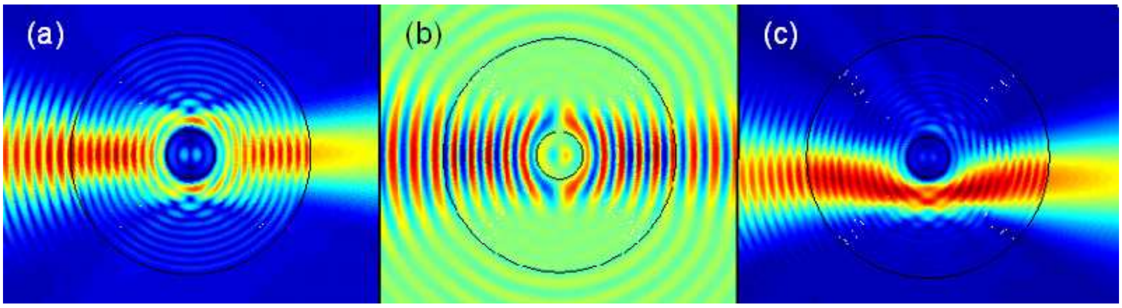

Fig. 4. (a) On-axis gaussian beam shows the isotropic scattering as well as penetration within the cloak (b) Real part of the on axis gaussian beam (c) Intensity in false color of a slightly off axis gaussian beam. The cloaking performance degrades if the impact parameter is not large compared to the beam width.

5. Conclusion

Starting with the family of ray trajectories that bend around a certain region in space we have derived the dielectric response required for non magnetic cloaking. Numerical wave simulations of gaussian beam scattering from the cloak agree with the ray calculations. An on-axis ray is not bent around the cloak in the ray theory and full wave simulations show that a gaussian beam striking the cloak with zero impact parameter penetrates the cloak and hence degrades the performance.

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